Critical collapse in 2+1 dimensional AdS spacetime: quasi-CSS solutions and linear perturbations

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Abstract

We construct a one-parameter family of exact time-dependent solutions to 2+1 gravity with a negative cosmological constant and a massless minimally coupled scalar field as source. These solutions present a continuously self-similar (CSS) behavior near the central singularity, as observed in critical collapse, and an asymptotically AdS behavior at spatial infinity. We consider the linear perturbation analysis in this background, and discuss the crucial question of boundary conditions. These are tested in the special case where the scalar field decouples and the linear perturbations describe exactly the small-mass static BTZ black hole. In the case of genuine scalar perturbations, we find a growing mode with a behavior characteristic of supercritical collapse, the spacelike singularity and apparent horizon appearing simultaneously and evolving towards the AdS boundary. Our boundary conditions lead to the value of the critical exponent $\gamma = 0.4$.

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1 Introduction

The exploration of possible connections between seemingly different fields of physics has always attracted much interest and curiosity. In this respect, the observation first made by Choptuik [1] that gravitational collapse does exhibit many of the features typical of critical phenomena has opened a new of line of research (see for instance the review paper [2] and references therein). Specifically, Choptuik showed numerically, on the example of a gravitating spherically symmetric massless scalar field, that certain field configurations lying at the black hole threshold in the phase space of initial data for the gravity-matter system are characterized by universality, power-law scaling of the black hole mass and self-similarity (discrete self-similarity in this specific case). Recently, Pretorius and Choptuik [3] have considered the same problem in a simplified context, namely the collapse of a massless and minimally coupled scalar field in 2+1 dimensional AdS spacetime. The inclusion of a negative cosmological constant is necessary for the existence of vacuum black hole solutions in 2+1 dimensions, the BTZ black holes [4]. By considering numerical collapsing configurations close to black hole formation, they obtained threshold solutions exhibiting characteristic critical features, namely power-law scaling and continuous self-similarity (CSS). Numerical calculations along similar lines were also performed by Husain and Olivier [5].

While the numerical investigation of critical collapse is by now a well-established branch of general relativity, far less is known on the purely analytical level. In 3+1 dimensions, a family of exact CSS solutions to the Einstein-massless scalar field model, first given by Roberts [6], was studied by various authors[7]-[10]. In particular, following the general analysis of linear perturbations of critical solutions [11, 12], Frolov studied the linear perturbations of the threshold Roberts solution, and determined the mass-scaling exponent. In 2+1 dimensions, Garfinkle [13] found a class of CSS solutions to the gravitating massless scalar field model with vanishing cosmological constant, and argued that one of these solutions should approximate the critical solution observed in [3] near the singularity. The drawback is that it is not possible to show that linear perturbations of such a solution lead to black hole formation, both because the solution of the perturbation problem necessitates boundary conditions which cannot be consistently stated for a solution which is known only in the vicinity of the singularity, and more fundamentally because black hole formation in 2+1 dimensions requires a cosmological constant, which does not appear in Garfinkle's CSS solutions. Subsequently, the present authors [14] found another CSS so-

lution to the same problem, and showed how to extend this solution to a quasi-CSS solution of the full problem with cosmological constant. It was then possible to perform the linear perturbation analysis of this extended solution, and to show that it did lead to black hole formation.

In this paper we generalize the work of [14] to a new one-parameter class of CSS solutions, and significantly refine the analysis of black-hole formation. In section 2 we review Garfinkle's one-parameter family of exact CSS solutions to the $\Lambda = 0$ equations of motion. The question of extending these to solutions of the full $\Lambda \neq 0$ system is addressed in section 3, where for a special parameter value we find an exact cosmological solution. Then, in sections 4 and 5, we use a limiting procedure to derive from Garfinkle's solutions a new class of $\Lambda = 0$ CSS solutions, discuss their properties, and carry out their extension to solutions for the case with negative cosmological constant. The main contribution of this paper is represented by sections 6, 7 and 8, where we consider the linear perturbation analysis in the background of these quasi-CSS solutions. The discussion of the most appropriate boundary conditions to be imposed on the perturbations, a separate analysis for the limiting case where the scalar field decouples, and a discussion of two possible scenarios for black hole formation in the linear approximation are then carried out. Finally, in section 9 we compare our results with those of previous numerical and analytical investigations.

2 First class of CSS solutions

The Einstein equations with a cosmological constant and a minimally coupled massless scalar field source are

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{2.1}$$

where $T_{\mu\nu}$ is the stress-energy tensor for the scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi\,, \tag{2.2}$$

the cosmological constant $\Lambda = -l^{-2}$ is negative for asymptoticaly AdS spacetime, and the gravitational constant κ is assumed to be positive.

The (2+1)-dimensional rotationally symmetric line element may be written in double null coordinates as:

$$ds^2 = e^{2\sigma}dudv - r^2d\theta^2, \tag{2.3}$$

with metric functions $\sigma(u, v)$ and $r(u, v)$. The Einstein equations (2.1) (where we take $\kappa = 1$) and the associated scalar field equation are

$$r_{,uv} = \frac{\Lambda}{2} r e^{2\sigma}, \tag{2.4}$$

$$2\sigma_{,uv} = \frac{\Lambda}{2} e^{2\sigma} - \phi_{,u}\phi_{,v}, \tag{2.5}$$

$$2\sigma_{,u} r_{,u} - r_{,uu} = r\phi_{,u}^2, \tag{2.6}$$

$$2\sigma_{,v} r_{,v} - r_{,vv} = r\phi_{,v}^2, \tag{2.7}$$

$$2r\phi_{,uv} + r_{,u}\phi_{,v} + r_{,v}\phi_{,u} = 0 \quad . \tag{2.8}$$

For a given solution of these equations, the Ricci scalar is

$$R = -6\Lambda + 4e^{-2\sigma}\phi_{,u}\phi_{,v}. \tag{2.9}$$

Assuming $\Lambda = 0$, Garfinkle has found [13] the following family of exact CSS solutions to these equations

$$\begin{aligned} ds^2 &= -A\left(\frac{(\sqrt{\hat{v}} + \sqrt{-\hat{u}})^4}{-\hat{u}\hat{v}}\right)^{c^2} d\hat{u}\, d\hat{v} - \frac{1}{4}(\hat{v} + \hat{u})^2\, d\theta^2\,, \\ \phi &= -2c\ln(\sqrt{\hat{v}} + \sqrt{-\hat{u}})\,, \end{aligned} \tag{2.10}$$

depending on an arbitrary constant c and a scale $A > 0$. These solutions are continuously self-similar with homothetic vector $(\hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}})$. An equivalent form of these CSS solutions, obtained by making the transformation

$$-\hat{u} = (-\bar{u})^{2q}\,, \quad \hat{v} = (\bar{v})^{2q} \qquad (1/2q = 1 - c^2) \tag{2.11}$$

to the barred null coordinates $(\bar{u}, \bar{v})$, is

$$\begin{aligned} ds^2 &= -\bar{A}(\bar{v}^q + (-\bar{u})^q)^{2(2q-1)/q}\, d\bar{u}\, d\bar{v} - \frac{1}{4}(\bar{v}^{2q} - (-\bar{u})^{2q})^2\, d\theta^2\,, \\ \phi &= -2c\ln(\bar{v}^q + (-\bar{u})^q)\,. \end{aligned} \tag{2.12}$$

Garfinkle suggested that the line element (2.10) describes critical collapse in the sector $r = -(\hat{u} + \hat{v})/2 \geq 0$, near the future point singularity $\hat{u} = \hat{v} = 0$, where the effect of the cosmological constant can be neglected. The global spacetime structure of depends on the value of the parameter c^2 and is discussed in [14]. As shown in [13], the metric (2.12) can be extended through the surface $\bar{v} = 0$ only for $q = n$, where n is a positive integer.

Let us mention here that these solutions belong to a larger class of solutions which may be generated from the Garfinkle solutions (2.10) or (2.12) by

the Ida-Morisawa transformation. Ida and Morisawa considered [15] three-dimensional Einstein-scalar fields with a hypersurface orthogonal spacelike Killing vector, parametrized by (2.3), and showed that the transformation

$$(r', \sigma', \phi') = (r, \sigma + b\phi + (b^2/2)\ln r, \phi + b\ln r) \tag{2.13}$$

generates a family (parametrized by b) of Einstein-scalar fields. Applied to (2.12), this transformation leads to the solutions

$$\begin{aligned} ds'^2 &= -\bar{A}'(\bar{v}^q + (-\bar{u})^q)^{(2c-b)^2}(-\bar{v}^q + (-\bar{u})^q)^{b^2}\, d\bar{u}\, d\bar{v} - \frac{1}{4}(\bar{v}^{2q} - (-\bar{u})^{2q})^2\, d\theta^2\,, \\ \phi' &= (b-2c)\ln(\bar{v}^q + (-\bar{u})^q) + b\ln(\bar{v}^q - (-\bar{u})^q)\,, \end{aligned} \tag{2.14}$$

with $q = 1/2(1-c^2)$. In the case $b = 2c$, this solution (which was previously given in [14], Eq. (2.16)) is again CSS. For $b \neq 0$, these solutions have a timelike central ($r = 0$) curvature singularity at $\bar{v}^q - (-\bar{u})^q = 0$. They can again be extended through the surface $\bar{v} = 0$ for $q = n$. In the case $q = n$ odd, the extension of a solution (2.14) with parameters $(\bar{A}, b, c)$ leads to a solution of the same class with parameters $(\bar{A}' = -\bar{A}, b' = b - 2c, c' = -c)$.

3 A class of quasi-CSS cosmologies

For a consistent analytical treatment of critical collapse, we should be able to extend the $\Lambda = 0$ CSS solutions (2.10) to $\Lambda < 0$ quasi-CSS solutions, i.e. to construct (explicitly or implicitly) solutions of the full field equations with the behavior (2.10) near the singularity. However, we have been unable to find an ansatz which separates the variables and reduces the field equations to ordinary differential equations, except in the special case $c^2 = 1/2$ ($q = 1$). In this case the solution (2.12) with $\bar{A} = 1$ may be written, after transforming to coordinates (R, T) defined by

$$\bar{u} = \frac{R - T}{\sqrt{2}}, \quad \bar{v} = \frac{R + T}{\sqrt{2}}, \tag{3.1}$$

in the form of a FRW cosmology with flat spatial sections,

$$\begin{aligned} ds^2 &= T^2(\, dT^2 - dR^2 - R^2\, d\theta^2), \\ \phi &= -\sqrt{2}\ln T. \end{aligned} \tag{3.2}$$

This form suggests searching for an extension by separating the equations in the variables R, T. With the metric parametrization

$$ds^2 = e^{2\Sigma}(dT^2 - dR^2) - r^2 d\theta^2, \tag{3.3}$$

these equations read

$$
\begin{aligned}
\ddot{r} - r'' &= 2\Lambda r e^{2\Sigma}, \\
2(\ddot{\Sigma} - \Sigma'') &= 2\Lambda e^{2\Sigma} - \dot{\phi}^2 + \phi'^2, \\
2(\dot{\Sigma}\dot{r} + \Sigma' r') - \ddot{r} - r'' &= r(\dot{\phi}^2 + \phi'^2), \\
\Sigma'\dot{r} + \dot{\Sigma} r' - \dot{r}' &= r\dot{\phi}\phi', \\
r(\ddot{\phi} - \phi'') + \dot{r}\dot{\phi} - r'\phi' &= 0,
\end{aligned}
\tag{3.4}
$$

with $\dot{} = \partial/\partial_T$ and $' = \partial/\partial_R$. Let us make the ansatz

$$\phi = A(T) + B(R), \quad r = F(T)G(R). \tag{3.5}$$

The equations separate and lead to a FRW cosmology if

$$G'' = kG \tag{3.6}$$

(k constant). One finds that for the solution to reduce to (3.2) for $\Lambda = 0$ one must have B constant, e.g. $B = 0$, the remaining equations being integrated by

$$
\begin{aligned}
&\dot{A} = \frac{\alpha}{F}, \quad e^{\Sigma} = \beta F, \\
&\dot{F}^2 - \alpha^2/2 - kF^2 - \Lambda\beta^2 F^4 = 0,
\end{aligned}
\tag{3.7}
$$

where α and β are integration constants; without loss of generality we may choose $\alpha = \sqrt{2}$ (by rescaling times) and $\beta = 1$. The resulting metric

$$ds^2 = F^2(T)(dT^2 - dR^2 - G^2(R)d\theta^2), \tag{3.8}$$

where $G(R)$ solves the equation $G'' = kG$ with the initial conditions $G(0) = 0$, $G'(0) = 1$ for regularity, is of the FRW type, and reduces to (3.2) in the limit $\Lambda \to 0$ if $k \to 0$ in this limit.

The discriminant $(k^2 - 4\Lambda)$ of the effective potential in the last equation (3.7) is positive definite for Λ negative, so that this potential has two roots $F_-^2 = \tau_- < 0$ and $F_+^2 = \tau_+ > 0$. It follows that this cosmology is time symmetric around the turning point $F^2 = \tau_+$, with initial and final singularities at $F^2 = 0$. Defining the new time variable $\tau \equiv F^2(T)$, the line element (3.8) may be rewritten as[1]

$$ds^2 = \frac{l^2}{4}\frac{d\tau^2}{(\tau - \tau_-)(\tau_+ - \tau)} - \tau(dR^2 + G^2(R)d\theta^2), \tag{3.9}$$

[1] For $G(R) = \text{const.}$ ($k = 0$, so that $\tau_- = -\tau_+$), this solution reduces to the cosmological version of the static solution (4.4) with $a = 0$ (put $\rho_- = -\rho_+ = -\tau_+$ and $\rho^2 = \tau_+^2 - \tau^2$).

$(\Lambda = -l^{-2})$ with

$$\begin{aligned} G(R) &= \mu^{-1} \sinh \mu R \quad (k = \mu^2 > 0), \\ G(R) &= R \quad (k = 0), \\ G(R) &= \nu^{-1} \sin \nu R \quad (k = -\nu^2 < 0). \end{aligned} \tag{3.10}$$

In the special case $k = 0$, this line element may be rewritten as

$$ds^2 = F^2 \left(\frac{dF^2}{1 - F^4/l^2} - dR^2 - R^2 d\theta^2 \right), \tag{3.11}$$

showing clearly the restoration of self-similarity in the limit $l \to \infty$.

4 Second class of CSS solutions

From the Garfinkle class of CSS solutions (2.10), we may derive a new class of CSS solutions by a generalization of the construction of [14]. The double null ansatz (2.3) is invariant under boosts $\hat{u} \to \alpha \hat{u}$, $\hat{v} \to \alpha^{-1} \hat{v}$. Rescaling angles by $\theta \to (2/\alpha)\theta$ and putting $A = \alpha^{-2c^2} \hat{A}$, we obtain from (2.10), in the limit of an infinite boost $\alpha \to \infty$,

$$\begin{aligned} ds^2 &= -\hat{A} (\frac{-\hat{u}}{\hat{v}})^{c^2} d\hat{u} d\hat{v} - \hat{u}^2 d\theta^2, \\ \phi &= -c \ln(-\hat{u}). \end{aligned} \tag{4.1}$$

These new solutions are obviously, as the original Garfinkle solutions, continuously self-similar with homothetic vector $(\hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}})$. A simpler form of these CSS solutions may be obtained by transforming to new null coordinates

$$u = -(-\hat{u})^{1+c^2}, \quad v = \hat{v}^{1-c^2}, \tag{4.2}$$

which yields (with the normalization $\hat{A} = c^4 - 1$)

$$\begin{aligned} ds^2 &= du dv - (-u)^{2/(1+c^2)} d\theta^2, \\ \phi &= -\frac{c}{1+c^2} \ln(-u). \end{aligned} \tag{4.3}$$

For $c^2 = 1$ we recover the "new CSS solution" of [14].

Interestingly, the CSS solutions (4.1) may also be obtained by submitting the singular static solutions of the cosmological Einstein-scalar field

equations to an infinite boost near the singularity. These solutions, derived in [16],

$$\begin{aligned} ds^2 &= A\,|\rho-\rho_+|^{1/2+a}\,|\rho-\rho_-|^{1/2-a}\,dt^2 - \frac{4|\Lambda|}{A}\,|\rho-\rho_+|^{1/2-a}\,|\rho-\rho_-|^{1/2+a}\,d\theta^2 \\ &\quad + \frac{d\rho^2}{4\Lambda(\rho-\rho_+)(\rho-\rho-)}\,, \qquad \phi = \sqrt{\frac{1-4a^2}{8}}\,\ln\left(\frac{|\rho-\rho_+|}{|\rho-\rho_-|}\right) \end{aligned} \tag{4.4}$$

(with $-1/2 \le a \le 1/2$, and $\rho_\pm = O(\Lambda^{-1})$) are singular for $\rho = \rho_\pm$. In the neighborhood of one of these singularities, e.g. ρ_+, the transformation $\rho = \rho_+ + x^{4/(1-2a)}$ leads (after appropriate rescalings) for $x \to 0$, or equivalently for $\Lambda \to 0$, to the exact $\Lambda = 0$ solution [17]

$$ds^2 = x^{2(1+2a)/(1-2a)}(dt^2 - dx^2) - x^2 d\theta^2, \quad \phi = \sqrt{\frac{2(1+2a)}{1-2a}}\,\ln x. \tag{4.5}$$

Transforming to null coordinates $\hat{u} = t - x$, $\hat{v} = t + x$, and boosting this solution by $\hat{u} \to \alpha\hat{u}$, $\hat{v} \to \alpha^{-1}\hat{v}$, we obtain in the limit of an infinite boost (again after appropriate rescalings)

$$ds^2 = (-\hat{u})^{2(1+2a)/(1-2a)}d\hat{u}d\hat{v} - \hat{u}^2 d\theta^2, \quad \phi = \sqrt{\frac{2(1+2a)}{1-2a}}\,\ln|\hat{u}|. \tag{4.6}$$

After a suitable transformation on the coordinate $\hat{v}$, this is found to coincide with the CSS solution (4.1) provided we identify

$$c^2 = 2\frac{1+2a}{1-2a}. \tag{4.7}$$

For $a = -1/2$ $(+1/2)$ the solution (4.4) is no longer singular for $\rho = \rho_+$ (ρ_-) but reduces to the BTZ black hole

$$ds^2 = (r^2/l^2 - M)dt^2 - \frac{dr^2}{(r^2/l^2 - M)} - r^2 d\theta^2, \tag{4.8}$$

with $\phi = 0$. Near $r = 0$, this reduces for $M > 0$ to the Minkowski cosmology $ds^2 = M^{-1}dr^2 - Mdt^2 - r^2d\theta^2$, which after an infinite boost leads to the metric (4.3) with $c = 0$,

$$ds^2 = dudv - u^2 d\theta^2. \tag{4.9}$$

The infinite boost limit corresponds physically to the massless limit, and so one would expect that the metric (4.9) can alternatively be derived from

(4.8) by taking the limit $M \to 0$ (this time without an infinite boost). Indeed, the extreme ($M = 0$) BTZ black hole, or "vacuum solution"

$$ds^2 = (r^2/l^2)dt^2 - \frac{dr^2}{r^2/l^2} - r^2 d\theta^2, \tag{4.10}$$

may be rewritten (with $x = l^2/r$) as

$$\begin{aligned} ds^2 &= \frac{l^2}{x^2}(dt^2 - dx^2 - l^2 d\theta^2) \\ &= \frac{4l^2}{(U-V)^2}(dU dV - l^2 d\theta^2) \end{aligned} \tag{4.11}$$

in null coordinates $U = t - x$, $V = t + x$. The transformation

$$U = -2l^2/u, \quad V = v/2 \tag{4.12}$$

transforms (4.11) to

$$ds^2 = \frac{1}{(1 + uv/4l^2)^2}(du dv - u^2 d\theta^2). \tag{4.13}$$

This goes over to the $c^2 = 0$ CSS solution (4.9) in the limit $\Lambda \to 0$ ($l \to \infty$) or, equivalently, in the near-horizon limit $r \to 0$ ($u \to 0$).

The $c^2 = 0$ spacetime (4.9) is Ricci-flat, and so (in 2+1 dimensions) is flat. Therefore it is locally equivalent to Minkowski spacetime. The explicit local transformation from the Minkowski metric written in the double null form

$$ds^2 = du\, dw - dy^2 \tag{4.14}$$

to the metric (4.9) is

$$u = u, \quad y = u\theta, \quad w = v + u\theta^2. \tag{4.15}$$

The rotationally symmetric $c^2 = 0$ CSS spacetime is obtained by periodically identifying θ with $\theta + 2\pi$. From this vacuum solution ($r = -u$, $\sigma = 0$, $\phi = 0$), the entire second class (4.1) of $c^2 \neq 0$ CSS solutions may be generated by the Ida-Morisawa transformation (2.13) with $b = c$.

For all $c \neq 0$, the CSS solutions (4.3) have a high degree of symmetry, with the four Killing vectors

$$\begin{aligned} L_1 &= v\partial_v - u\partial_u + \frac{1}{c^2+1}\,\theta\partial_\theta\,, \\ L_2 &= -2\frac{c^2-1}{c^2+1}\,\theta\partial_v + (-u)^{\frac{c^2-1}{c^2+1}}\,\partial_\theta\,, \\ L_3 &= \partial_v\,, \\ L_4 &= \partial_\theta\,. \end{aligned} \tag{4.16}$$

For $c^2 = 1$, the above form of L_2 is replaced by

$$L_2 = \theta\partial_v - \frac{1}{2}\ln(-u)\partial_\theta. \tag{4.17}$$

As in the case $c^2 = 1$ [14], the Lie algebra generated by the Killing vectors (4.16) is solvable. In the case $c^2 = 0$, the flat metric (4.9) admits of course six Killing vectors generating the (2+1)-dimensional Poincaré algebra.

Again as in the case $c^2 = 1$, the metric (4.3) is devoid of scalar curvature singularities. The nature of the corresponding geometry may be understood from the study of geodesic motion in this spacetime. The geodesic equations are integrated by

$$\dot{u} = \pi\,, \quad (-u)^{2/(c^2+1)}\dot{\theta} = -l\,, \quad \pi\dot{v} + l\dot{\theta} = \varepsilon\,, \tag{4.18}$$

where π, l and ε are constants of the motion. The spacetime is extendible to a geodesically complete spacetime in two cases:

a) $c^2 = 0$. This is the near-horizon limit (4.9) of the BTZ vacuum (4.13), and is diffeomorphic to the Minkowski metric, as explained above.

b) $c^2 \to \infty$. In this limit the scalar field decouples and the metric (4.3) reduces to the geodesically complete cylindrical Minkowski metric.

In the other cases, the third equation (4.18) integrates to

$$v = \frac{\varepsilon}{\pi^2}u - \frac{l^2(c^2+1)}{\pi^2(c^2-1)}(-u)^{(c^2-1)/(c^2+1)} + const.\,, \tag{4.19}$$

showing that for $c^2 < 1$ nonradial geodesics terminate at $u = 0, v \to +\infty$, while radial geodesics ($l = 0$) can be continued through the null line $u = 0$, which they cross at finite v, to $u \to +\infty$, only the endpoint $v \to +\infty$ of the null line $u = 0$ being singular. In the case $c^2 = 1$ (treated in [14]), (4.18) is replaced by

$$v = \frac{\varepsilon}{\pi^2}u - \frac{l^2}{\pi^2}\ln(-u) + const.\,, \tag{4.20}$$

leading to the same conclusion as for $c^2 < 1$. On the other hand, for $c^2 > 1$ nonradial geodesics terminate at $u = 0$, $v =$ const., and the whole line $u = 0$ corresponds to a singularity of the geometry. So the $c^2 = 1$ solution is an extreme solution in the manifold of CSS solutions (4.3), lying at the threshold between two classes of solutions ($0 < c^2 \leq 1$ and $c^2 > 1$) differing by their global spacetime geometry (Fig. 1).

5 Quasi-CSS extensions of the second class

In this section we proceed to extend the second class of $\Lambda = 0$ CSS solutions (4.3) to exact solutions of the full $\Lambda < 0$ equations (the self-similarity being then broken by the cosmological constant). Following [14] we make the ansatz

$$ds^2 = e^{2\nu(x)} du dv - (-u)^{\frac{2}{c^2+1}} \rho^2(x) d\theta^2, \quad \phi = -\frac{c}{c^2+1} \ln|u| + \psi(x)\,, \quad (5.1)$$

with $x = uv$. This reduces to the CSS solution (4.3) for $\rho = 1$, $\nu = \psi = 0$, and preserves the Killing subalgebra (L_1, L_4). Inserting this ansatz into the field equations (2.4)-(2.8) leads to the system

$$\begin{aligned}
x\rho'' + \frac{c^2+2}{c^2+1}\rho' &= \frac{\Lambda}{2}\rho e^{2\nu}, && (5.2)\\
2(x\nu'' + \nu') + \psi'(x\psi' - \frac{c}{c^2+1}) &= \frac{\Lambda}{2} e^{2\nu}, && (5.3)\\
x^2(-\rho'' + 2\rho'\nu' - \rho\psi'^2) + \frac{2}{c^2+1} x(-\rho' + \rho(\nu' + c\psi')) &= 0 && (5.4)\\
-\rho'' + 2\rho'\nu' - \rho\psi'^2 &= 0 && (5.5)\\
2x(\rho\psi')' + \frac{2c^2+3}{c^2+1}\rho\psi' &= \frac{c}{c^2+1}\rho' && (5.6)
\end{aligned}$$

($' = d/dx$). A simple first integral, obtained by comparing (5.4) and (5.5), is

$$\rho = e^{\nu + c\psi} \quad (5.7)$$

(with the integration constant set to 1 by the boundary conditions (5.8)).

The boundary conditions

$$\rho(0) = 1, \quad \nu(0) = 0, \quad \psi(0) = 0 \quad (5.8)$$

lead to a unique solution reducing to (4.3) near $u = 0$. The small x behavior of this solution is

$$\begin{aligned}
\rho &\simeq 1 + \frac{c^2+1}{2(c^2+2)}\Lambda x, \\
\nu &\simeq \frac{(c^2+1)(c^2+3)}{2(c^2+2)(2c^2+3)}\Lambda x, \\
\psi &\simeq \frac{c(c^2+1)}{2(c^2+2)(2c^2+3)}\Lambda x.
\end{aligned} \quad (5.9)$$

The analysis of [14] can be straightforwardly extended here, leading to the conclusion that when x decreases from $x = 0$, the functions ρ and $e^{2\nu}$ increase indefinitely, going to infinity for a finite value $x = x_1$. The numerical solution of the system (5.2)-(5.6) with boundary conditions (5.8) leads to values $x_1(c^2)$ equal to $-4l^2$ for $c^2 = 0$ and for $c^2 \to \infty$, and remaining of the order of $-4l^2$ for finite c^2, the maximum value being $x_1(1) = -3.876l^2$. The behavior of the metric functions near x_1 is found to be

$$\begin{aligned}
\rho &= \rho_1 \left(\frac{1}{\bar{x}} + \frac{c^2}{2(c^2+1)x_1} - \frac{c^2 \bar{x} \ln(\bar{x})}{12(c^2+1)^2 x_1^2} + ... \right) \\
e^{2\nu} &= \frac{4x_1}{\Lambda \bar{x}^2} \left(1 + \frac{c^2 \bar{x}^2 \ln(\bar{x})}{12(c^2+1)^2 x_1^2} + ... \right) \\
\psi &= \psi_1 + \frac{c\bar{x}}{2(c^2+1)x_1} - \frac{c\bar{x}^2}{8(c^2+1)^2 x_1^2} \ln(\bar{x}) + ...
\end{aligned} \tag{5.10}$$

($\bar{x} = x - x_1$). Changing to coordinates $(\bar{T}, \bar{R})$ defined by

$$u = -l^{c^2+1} e^{\bar{R}-\bar{T}}, \quad v = l^{1-c^2} e^{\bar{R}+\bar{T}}, \tag{5.11}$$

we obtain from (5.10) the leading asymptotic behaviors near $\bar{R} = \bar{R}_1$

$$ds^2 \simeq \frac{l^2}{(\bar{R}_1 - \bar{R})^2} (d\bar{T}^2 - d\bar{R}^2 - e^{\frac{2}{c^2+1}(\bar{T}_1 - \bar{T})} d\theta^2) , \quad \phi = \phi_1 + c\bar{T}/(c^2+1) \tag{5.12}$$

($\bar{R} - \bar{R}_1 \simeq \bar{x}/2x_1$). Making the further coordinate transformation,

$$\bar{R} - \bar{R}_1 = -(c^2+1)/XT \,, \quad \bar{T} - \bar{T}_1 = (c^2+1) \ln(T/(c^2+1)) \,, \tag{5.13}$$

we arrive at the asymptotic form

$$ds^2 \simeq l^2 \left(X^2 dT^2 - \frac{dX^2}{X^2} - X^2 d\theta^2 \right) , \quad \phi = \phi_1 + c \ln T \,, \tag{5.14}$$

showing that the metric is asymptotically AdS (with logarithmic subdominant terms). The corresponding conformal diagrams for the cases $0 < c^2 \leq 1$ and $c^2 > 1$ are shown in Fig. 2.

For $c^2 = 0$ or $c^2 \to \infty$, the solution of Eq. (5.6) is $\rho\psi' = \alpha x^{-p}$, with α constant and $p = 3/2$ for $c^2 = 0$ or $p = 1$ for $c^2 \to \infty$, so that the boundary conditions (5.8) enforce $\alpha = 0$, i.e. $\psi = 0$, and the extended metric (5.1) is a vacuum metric. For $c^2 = 0$ this metric is

$$ds^2 = \frac{1}{(1 + uv/4l^2)^2} (dudv - u^2 d\theta^2), \tag{5.15}$$

which we recognize as the BTZ vacuum (4.13). For $c^2 \to \infty$, the extended metric [2]

$$ds^2 = \frac{1}{(1 + uv/4l^2)^2}(dudv - (1 - uv/4l^2)^2 d\theta^2), \tag{5.16}$$

is obtained from the BTZ metric (4.8) by transforming first to coordinates (R, T) with

$$r = M^{1/2} l \coth(M^{1/2} R/l), \quad t = T, \tag{5.17}$$

which leads to

$$ds^2 = \frac{M}{\sinh^2(M^{1/2}R/l)}(dT^2 - dR^2) - Ml^2 \coth^2(M^{1/2}R/l)\, d\theta^2, \tag{5.18}$$

then, choosing the convenient normalization $Ml^2 = 1$, to coordinates (u, v) defined by

$$u = -2l\, e^{(R-T)/l^2}, \quad v = 2l\, e^{(R+T)/l^2}. \tag{5.19}$$

So the family of time-dependent quasi-CSS solutions (5.1) interpolates between the BTZ vacuum for $c^2 = 0$ and the massive BTZ metric for $c^2 \to \infty$.

6 Perturbations: general setup

Now we study linear perturbations of the quasi-CSS solutions (5.1) along the lines of the analysis of [9, 10, 14]. The relevant parameter in critical collapse being the retarded time [13] $T = -\ln(-\hat{u}) = -(1/(c^2+1))\ln(-u)$, we expand these perturbations in modes proportional to $e^{kT} = (-u)^{-k/(c^2+1)}$, with k a complex constant. Only those modes with $Re(k) > 0$, which grow when $T \to +\infty$, will possibly lead to black hole formation. Keeping only one mode, we decompose the perturbed fields as

$$\begin{aligned} r &= (-u)^{1/(c^2+1)}(\rho(x) + (-u)^{-k/(c^2+1)}\tilde{r}(x)), \\ \phi &= -\frac{c}{c^2+1}\ln|u| + \psi(x) + (-u)^{-k/(c^2+1)}\tilde{\phi}(x), \\ \sigma &= \nu(x) + (-u)^{-k/(c^2+1)}\tilde{\sigma}(x). \end{aligned} \tag{6.1}$$

The linearization of the Einstein equations (2.4)-(2.8) in the perturbations $\tilde{r}$, $\tilde{\phi}$, $\tilde{\sigma}$ leads to the system

$$x\tilde{r}'' + \frac{c^2+2-k}{c^2+1}\,\tilde{r}' = \frac{\Lambda}{2}\, e^{2\nu}(\tilde{r} + 2\rho\tilde{\sigma}), \tag{6.2}$$

[2] In this case the derivation of (5.7) breaks down. The solution (5.16) is obtained by first integrating (5.5) to $\rho' = \text{const.}\ e^{2\nu}$, then integrating (5.2) with the boundary conditions (5.8).

$$2x\tilde{\sigma}'' + \frac{2(c^2+1-k)}{c^2+1}\,\tilde{\sigma}' = \Lambda e^{2\nu}\tilde{\sigma} - \left(2x\psi' - \frac{c}{c^2+1}\right)\tilde{\phi}' + \frac{k}{c^2+1}\,\psi'\tilde{\phi}, \tag{6.3}$$

$$-(1-k)x\tilde{r}' + \left((1-k)x\nu' + \frac{k(1-k-c^2)}{2(c^2+1)}\right)\tilde{r} + \rho x\tilde{\sigma}' - k\left(x\rho' + \frac{\rho}{c^2+1}\right)\tilde{\sigma} =$$
$$-\rho\left(cx\tilde{\phi}' - k\left(\frac{c}{c^2+1} - x\psi'\right)\tilde{\phi}\right) - cx\psi'\tilde{r}, \tag{6.4}$$

$$2(\rho'\tilde{\sigma}' + \nu'\tilde{r}') - \tilde{r}'' = \psi'(2\rho\tilde{\phi}' + \psi'\tilde{r}), \tag{6.5}$$

$$2x\rho\tilde{\phi}'' + \left(2x\rho' + \frac{2c^2+3-2k}{c^2+1}\,\rho\right)\tilde{\phi}' - \frac{k}{c^2+1}\,\rho'\tilde{\phi} + \left(2x\psi' - \frac{c}{c^2+1}\right)\tilde{r}'$$
$$+\left(2x\psi'' + \frac{2c^2+3-k}{c^2+1}\,\psi'\right)\tilde{r} = 0. \tag{6.6}$$

This linear differential system is of order four (Eqs. (6.4) and (6.5) are constraints, while the perturbed scalar field equation (6.6) is a consequence of the perturbed Einstein equations). An exact solution of the system (6.2)-(6.6) is given by

$$\begin{aligned}
\tilde{r}_k(x) &= \alpha(-x)^{1+k/(c^2+1)}\rho'(x)\,, \\
\tilde{\phi}_k(x) &= \alpha(-x)^{1+k/(c^2+1)}\psi'(x)\,, \\
\tilde{\sigma}_k(x) &= \alpha\left((-x)^{1+k/(c^2+1)}\nu'(x) - \frac{c^2+1+k}{2(c^2+1)}(-x)^{k/(c^2+1)}\right).
\end{aligned} \tag{6.7}$$

The corresponding first-order perturbed fields (6.1) are generated from the unperturbed fields by the gauge transformation $v \to v - \alpha v^{1+k/(c^2+1)}$. So, up to gauge transformations, the general solution of this system depends only on three integration constants, which should be determined by enforcing appropriate boundary conditions.

At this point we should emphasize that the question of which boundary conditions are truly "appropriate" is rather subtle in the general-relativistic context (for instance, in the related problem of scalar field collapse in 3+1 dimensions, different boundary conditions have been chosen in [9] and [10]), yet crucial for a proper determination of the eigenmodes.

The three boundaries of our quasi-CSS solutions are $u = 0$, $v = 0$ (or $x = 0$), and $x = x_1$ (the AdS boundary). At the boundary $u = 0$, corresponding to the original centre of the unperturbed quasi-CSS solution, it seems natural to impose that (A) the perturbed metric component r does not diverge too quickly, which would conflict with the linear approximation [10]. However this is subject to some ambiguity. For instance, should one consider the

limit $u \to 0$ with v held fixed, or rather with $x = uv$ held fixed? Also, the perturbed spacetime may develop an apparent horizon hiding the singularity at $u = 0$. In such a case it may seem preferable to view this apparent horizon, rather than the line $u = 0$, as the natural boundary of the perturbed spacetime on which to impose a boundary condition.

On the second boundary $v = 0$, it seems natural to require that (B1) the perturbed solution matches smoothly the original quasi-CSS solution [9]

$$\tilde{r}(0) = 0, \quad \tilde{\sigma}(0) = 0, \quad \tilde{\phi}(0) = 0 \tag{6.8}$$

(on account of (5.8), the perturbed solution then also matches smoothly the original CSS solution). However conditions (6.8) do not ensure that the curvature tensor remains finite on $v = 0$, which seems essential for a smooth matching. This last requirement is automatically satisfied if the "weak" conditions (6.8) are supplemented with the the "strong" condition (B2) that the perturbations be (as the unperturbed quasi-CSS solution) analytic in v [14].

Finally, we shall require that (C) the perturbations grow slowly (i.e. not faster than the original fields) as the third boundary $x = x_1$ is approached [9], this condition ensuring the validity of the linear approximation near the AdS boundary. Note that, owing to the gauge freedom, it is only necessary that these various boundary conditions be satisfied up to gauge transformations, i.e. up to the addition of a gauge perturbation (6.7).

To enforce boundary conditions on the boundary $v = 0$, we must first investigate the behavior of the solutions of the linearized system near the boundary $x = 0$. Generalizing the analysis of [14], we assume the power-law behavior

$$\tilde{r}(x) \propto (-x)^p \tag{6.9}$$

(p constant). For the purpose of the determination of the exponent p, Eqs. (6.2), (6.3) and (6.5) can be approximated near $x = 0$ as

$$x\tilde{r}'' + \frac{c^2 + 2 - k}{c^2 + 1}\,\tilde{r}' \simeq \Lambda\tilde{\sigma}, \tag{6.10}$$

$$x\tilde{\sigma}'' + \frac{c^2 + 1 - k}{c^2 + 1}\,\tilde{\sigma}' \simeq \frac{c}{2(c^2 + 1)}\,\tilde{\phi}' \tag{6.11}$$

$$\frac{c^2 + 1}{c^2 + 2}\,\Lambda\tilde{\sigma}' - \tilde{r}'' \simeq \frac{c(c^2 + 1)}{(c^2 + 2)(2c^2 + 3)}\,\Lambda\tilde{\phi}' \tag{6.12}$$

(in (6.12) we have replaced the unperturbed fields by their small x behaviors (5.9)). The discussion of this system is simpler in the special (vacuum) case

$c^2 = 0$, which we shall consider in the next section, leaving the discussion of the general (scalar) case $c^2 \neq 0$ to Sect. 8.

7 BTZ black holes as vacuum perturbations

In the case $c^2 = 0$, we have seen that the quasi-CSS solution is the BTZ vacuum, with $\psi = 0$. Then, the linearized equations (6.2)-(6.6) decouple into the linearized sourceless Einstein equations and the linearized scalar field equation on the BTZ vacuum background. The solution of the three-dimensional cosmological Einstein equations being essentially unique, up to diffeomorphisms, the three-parameter family of perturbations of the BTZ vacuum must include the small-mass BTZ black holes, with the mass as perturbation parameter (amplitude), as we now check explicitly.

When $c^2 = 0$, Eqs. (6.11) and (6.12) decouple and must be satisfied separately. Then $\tilde{\sigma}$ can be eliminated between (6.10) and (6.12), leading to the third order equation

$$x\tilde{r}''' + (1-k)\tilde{r}'' \simeq 0, \tag{7.1}$$

which implies the secular equation

$$p(p-1)(p-1-k) = 0. \tag{7.2}$$

We note that the root p = 1+k corresponds to the gauge mode (6.7). Discarding this mode, we obtain from the full equations (6.2)-(6.6) the behavior of the perturbations near $x = 0$ in terms of two integration constants A, B:

$$\tilde{r}(x) \sim A + B(-x), \tag{7.3}$$

$$\tilde{\sigma}(x) \sim -\frac{A}{2} - (2-k)\Lambda^{-1}B + O(x). \tag{7.4}$$

This can only satisfy the boundary conditions (6.8) at $x = 0$ if

$$A = 0, \quad k = 2. \tag{7.5}$$

It follows that in this case the solution of the linearized system (6.2)-(6.5) with the initial conditions (6.8) is, up to a gauge transformation, unique. We now show that this solution is simply a linearization of the BTZ metric.

The BTZ metric (5.18) may be written in double-null form as

$$ds^2 = \frac{4l^2\, dU\, dV}{\sinh^2(U-V)} - Ml^2 \coth^2(U-V)\, d\theta^2, \tag{7.6}$$

with $U = \mu(T+R)$, $V = \mu(T-R)$, and $\mu = M^{1/2}/2l$. This may again be rewritten, in terms of the new null coordinates

$$u = -2\mu l^2 \coth U, \quad v = (2/\mu)\tanh V, \tag{7.7}$$

as

$$\begin{aligned} ds^2 &= \frac{\sinh^2 U \cosh^2 V}{\sinh^2(U-V)}\, du\, dv - 4\mu^2 l^4 \coth^2(U-V)\, d\theta^2 \\ &= \frac{1}{(1+uv/4l^2)^2}(dudv - (u+(M/4)v)^2 d\theta^2). \end{aligned} \tag{7.8}$$

This form of the BTZ metric again reduces to the vacuum CSS metric (4.9) in the infinite boost limit (which amounts to taking $v \to 0$) and, furthermore, can obviously be considered for small masses as a perturbation of the BTZ vacuum (4.13). From (7.8) the metric function r is

$$\begin{aligned} r &= \rho\,(-u-(M/4)v) \\ &= -u\,(\rho + (M/4)u^{-2}x\rho), \end{aligned} \tag{7.9}$$

with $x = uv$, and $\rho(x) = (1+x/4l^2)^{-1}$. This is of the form of (6.1) for the eigenmode $k=2$, as found above. On the other hand, the metric function $e^{2\sigma}$ in (7.8) does not depend on the perturbation parameter $M/4$, so that the perturbations $\tilde{r}$ and $\tilde{\sigma}$ are simply given by

$$\tilde{r} = \frac{M}{4}x\rho, \quad \tilde{\sigma} = 0, \tag{7.10}$$

which satisfy the condition of slow growth at the AdS boundary $x \to -4l^2$ (where $\tilde{r} \sim -Ml^2\rho$), as well as the requirement of finiteness of r for $u \to 0$ with v held fixed (but not for $u \to 0$ with x held fixed), and are easily checked to solve exactly the vacuum system (6.2)-(6.5).

It is instructive to analyze more closely in this (obviously very special) case the phenomenon of black hole formation. First we note that the unperturbed metric (4.13) can be extended beyond $u = -\infty$ by the coordinate transformation $u = -\bar{u}^{-1}$, leading to the metric

$$ds^2 = \frac{1}{(\bar{u} - v/4l^2)^2}(d\bar{u}dv - d\theta^2), \tag{7.11}$$

which is regular in the triangle $\bar{u} - v/4l^2 > 0$, the other boundaries of this triangle being the apparent horizons $r_{,v} = 0$ ($u = 0$) and $r_{,u} = 0$ ($v = -\infty$) through which the metric can be periodically extended to the

geodesically complete BTZ vacuum spacetime (Fig. 3). Bearing in mind that the perturbed metric (7.9) can similarly be extended beyond $u = -\infty$ and $v = -\infty$, and is obviously extendible to $u > 0$, we find that the effect of the perturbation is twofold (Fig. 4). First, the two apparent null central singularities $u = 0$ and $v = -\infty$ are replaced by two genuine spacelike central causal singularities [4] $u+(M/4)v = 0$. Second, the two null apparent horizons, displaced to

$$\begin{aligned} r_{,v} &= -u^2(\rho' + (M/4)u^{-2}(x\rho)') = \rho^2(\frac{u^2}{4l^2} - \frac{M}{4}) = 0, \\ r_{,u} &= -(\rho + x\rho' + (M/4)u^{-2}(-x\rho + x(x\rho)')) = -\rho^2(1 - \frac{Mv^2}{16l^2}) = 0, \end{aligned} \tag{7.12}$$

effectively shield the final and initial singularities.

While the situation here is static, rather than dynamical, this formation of apparent horizons in the linearized approximation (here exact) is formally similar to that observed in critical collapse. Recall that near-critical collapse is characterized by a critical exponent γ, defined by the scaling relation

$$Q \propto |p - p^*|^{s\gamma}, \tag{7.13}$$

for a quantity Q with dimension s depending on a parameter p (with $p = p^*$ for the critical solution). In the present case, it has been shown [18] that the BTZ vacuum is a critical point of the spinless BTZ black hole, and that the phase transition from extremal ($M = 0$) to nonextremal BTZ black holes is second order, so that a similar scaling relation applies. Choosing for Q the radius

$$r_{AH} = M^{1/2}l \tag{7.14}$$

of the apparent horizon ($s = 1$), and identifying $p-p^*$ with the perturbation amplitude $M/4$, we obtain from (7.14) the value of the BTZ critical exponent

$$\gamma = 1/2, \tag{7.15}$$

in agreement with the value derived previously either from the dynamical analysis of black-hole formation in 2+1 dimensions from spherical dust collapse [19] or point particle collisions [20], or from the consideration of phase transitions from non-extremal to extremal black holes [18] or from static to rotating black holes [21].

8 Scalar perturbations and black hole formation

Let us return to the case $c^2 \neq 0$ where the scalar field does not decouple. Eliminating now the functions $\tilde{\sigma}$ and $\tilde{\phi}$ between the three equations (6.10)-(6.12), we arrive at the fourth-order equation

$$x^2 \tilde{r}'''' - \frac{2k - 3c^2 - 7/2}{c^2+1} x\tilde{r}''' + \frac{(k-c^2-1)(k-c^2-3/2)}{(c^2+1)^2}\tilde{r}'' \simeq 0, \qquad (8.1)$$

which leads to the secular equation

$$p(p-1)\left(p - 1 - \frac{k}{c^2+1}\right)\left(p - \frac{k+c^2+1/2}{c^2+1}\right) = 0. \qquad (8.2)$$

Again, the root $p = 1 + k/(c^2+1)$ corresponds to the gauge mode (6.7). Discarding this mode, we obtain the general behavior of the perturbations near $x = 0$ in terms of three integration constants A, B, C:

$$\tilde{r}(x) \sim A + B(-x) + \Lambda C(-x)^{\frac{k+c^2+1/2}{c^2+1}}, \qquad (8.3)$$

$$\tilde{\sigma}(x) \sim -\frac{A}{2} - \frac{c^2+2-k}{c^2+1}\Lambda^{-1}B + O(x) - \frac{(c^2+3/2)(k+c^2+1/2)}{(c^2+1)^2} C(-x)^{\frac{k-1/2}{c^2+1}}, \qquad (8.4)$$

$$\tilde{\phi}(x) \sim \frac{(2-k-c^2)}{2c}A + \frac{c^2+2-k}{c(c^2+1)}\Lambda^{-1}B + O(x) + \frac{(c^2+3/2)(k+c^2+1/2)}{c(c^2+1)^2} C(-x)^{\frac{k-1/2}{c^2+1}}, \qquad (8.5)$$

for $k \neq 1/2$ (in the degenerate case $k = 1/2$, the power law $(-x)^{\frac{k+c^2+1/2}{c^2+1}}$ is replaced by a logarithm).

From these behaviors, we find that the smooth matching conditions (B1) at $v = 0$ are satisfied if either

$$A = 0, \quad B = 0 \quad (Re(k) > 1/2), \qquad (8.6)$$

or

$$A = 0, \quad k = c^2 + 2. \qquad (8.7)$$

The stronger condition (B2) of analyticity in v then implies, in the first case, that $k = n(c^2+1) + 1/2$, where n is a positive integer, and, in the

second case, that $C = 0$. So the boundary conditions (B1) and (B2) have two solutions:

$$a) \quad A = B = 0, \qquad k = n(c^2+1) + 1/2, \tag{8.8}$$

$$b) \quad A = C = 0, \qquad k = c^2 + 2. \tag{8.9}$$

When $u \to 0$ with v held fixed, the corresponding metric perturbations $(-u)^{(1-k)/(c^2+1)}\tilde{r}$ go to zero (in case a)) or to a finite limit (in case b)), so that in all cases the perturbed r has a finite limit and the boundary condition (A) is automatically satisfied. If on the other hand one takes the limit $u \to 0$ with x held fixed, then in all cases r diverges as $(-u)^{(1-k)/(c^2+1)}$. In case b) the divergence is the same (of the order $(-u)^{-1}$) as in the BTZ case, while it is weaker in case a) with $n = 1$, and stronger in case a) with $n \geq 2$. So a boundary condition that r does not diverge more than in the BTZ case when $u \to 0$ with x held fixed would select the eigenvalues $k = c^2 + 3/2$ (case a) with $n = 1$), or $k = c^2 + 2$ (case b)).

Two of the three integration constants A, B, C being now fixed, the first-order perturbation is now completely fixed up to an arbitrary scale, and (as discussed after (6.7)) within a gauge transformation. So we now must check that the natural condition (C) at the AdS boundary is indeed satisfied up to a gauge transformation. The leading behaviour of the background at the AdS boundary ($x \to x_1$) is, from Eqs. (5.10),

$$\rho \simeq \frac{\rho_1}{x - x_1}\,, \quad e^{2\nu} \simeq \left(\frac{4x_1}{\Lambda}\right)\frac{1}{(x-x_1)^2}\,, \quad \psi \simeq \psi_1\,. \tag{8.10}$$

We again assume a power-law behavior

$$\tilde{\sigma} \sim b\bar{x}^q \tag{8.11}$$

($\bar{x} = x - x_1$). Then Eq. (6.3), where $\tilde{\phi}$ can in first approximation be neglected, gives

$$q(q-1) = 2, \tag{8.12}$$

i.e. $q = -1$ or $q = 2$. Accordingly, Eq. (6.2) reduces near $\bar{x} = 0$ to

$$\tilde{r}'' - 2\bar{x}^{-2}\tilde{r} \simeq 4b\rho_1\bar{x}^{q-3}. \tag{8.13}$$

If $q = -1$, the behavior of the solution is governed by the right-hand side, i.e.

$$\tilde{r} \sim b\rho_1\bar{x}^{-2}, \tag{8.14}$$

which apparently violates the boundary condition (C) for $x \to x_1$. However the behaviors (8.11) with $q = -1$ and (8.14) are precisely those of the gauge

mode (6.7) near the AdS boundary, so that they can be transformed away by a gauge transformation, i.e. we can find a gauge[3] in which the behavior (8.14) is replaced by

$$\tilde{r} \sim \frac{E\rho_1}{x - x_1}, \tag{8.15}$$

(E constant) which is consistent with the boundary condition (C), and is an admissible small perturbation if its amplitude is small enough, $E \ll 1$. A finer analysis, where $\tilde{\phi}$ is not neglected in (6.3), then leads to the behavior $\tilde{\sigma} \propto \bar{x}^2 \ln|\bar{x}|$, and to the behavior (8.15) for $\tilde{r}$.

Black hole formation is characterized by the appearance of a central singularity hidden by an apparent horizon. The definition of these notions can be quite ambiguous in linearized perturbation theory. For our present purpose we will consider, rather than the centre $r = 0$, the coordinate singularity of the perturbed metric

$$\sqrt{g} = e^{2\sigma} r = (-u)^{\frac{1}{c^2+1}} e^{2\nu(x)} [\rho(x) + (-u)^{-\frac{k}{c^2+1}} (\tilde{r}(x) + 2\rho(x)\tilde{\sigma}(x))] = 0. \tag{8.16}$$

Let us discuss the solution of this equation near $x = 0$, considering separately the two cases a) and b). In case a) ($A = B = 0$), we see from Eqs. (8.3) and (8.4) that $\tilde{r} + 2\rho\tilde{\sigma}$ is dominated by $2\tilde{\sigma}$, so that the coordinate singularity develops for

$$(-u)^{\frac{k}{c^2+1}} = -2\tilde{\sigma} \simeq 2\frac{(c^2 + 3/2)(k + c^2 + 1/2)}{(c^2+1)^2} C(-x)^{\frac{k-1/2}{c^2+1}}, \tag{8.17}$$

which may be rewritten as

$$(-u)^{\frac{1}{2(c^2+1)}} \propto Cv^n. \tag{8.18}$$

For the lowest value $n = 1$ ($k = c^2 + 3/2$), we see that if $C > 0$ there is (to the order considered) no coordinate singularity for $v < 0$, while a spacelike coordinate singularity develops for $v > 0$. Note that the consideration of the centre $r = 0$ would have given quite different results, namely an eternal timelike centre at $(-u)^{\frac{k}{c^2+1}} = -\tilde{r} = -\Lambda C x^2$ (for $n = 1$). The location of the trapping horizon is given by

$$r_{,v} = (-u)^{\frac{c^2+2}{c^2+1}} (\rho' + (-u)^{-\frac{k}{c^2+1}} \tilde{r}') = 0, \tag{8.19}$$

[3] This gauge transformation is equivalent to a shift in the position x_1 of the AdS boundary.

or

$$(-u)^{\frac{k}{c^2+1}} \simeq 2\frac{(c^2+2)(k+c^2+1/2)}{(c^2+1)^2}C(-x)^{\frac{k-1/2}{c^2+1}} \tag{8.20}$$

near $x = 0$. For $n = 1$, $C > 0$, this spacelike trapping horizon develops for $v > 0$, just as the spacelike coordinate singularity (8.17) which it shields for all c^2 (Fig. 5). While we are unable to show the formation of a curvature singularity without a nonperturbative analysis, it is suggestive that this formation of a shielded coordinate singularity is precisely what would be expected for a genuine curvature singularity. As the AdS boundary $x = x_1$ is approached, it follows from the asymptotic behavior $\tilde{r} \sim E\rho + F$ that both the coordinate singularity and the trapping horizon approach the AdS boundary for a constant value of u, with a finite limit for the radius of the apparent horizon

$$r_{AH} = \left(-\frac{\rho'}{\tilde{r}'}\right)^{1/k}\left(\rho - \rho'\frac{\tilde{r}}{\tilde{r}'}\right) \quad \to \left(\frac{F}{E}\right)E^{1/k}. \tag{8.21}$$

Note that it is essential for this trapping horizon to form that the cosmological constant be non-zero. For $\Lambda = 0$, Eq. (8.17) for the coordinate singularity becomes exact, while on the other hand $\rho = 1$ and $\tilde{r} = 0$ everywhere so that $r_{,v}$ is identically zero.

In case b) ($A = C = 0$, $k = c^2+2$), a finer analysis leads to the behaviours (consistent with the exact solution (7.10)) near $x = 0$

$$\tilde{r} \sim -B(x + \frac{\Lambda}{4}(1+c^2)x^2 + \cdots), \tag{8.22}$$

$$\tilde{\sigma} \sim -B(\frac{c^2}{2}x + \cdots), \tag{8.23}$$

$$\tilde{\phi} \sim B(cx + \cdots). \tag{8.24}$$

From these, it follows that that if $B < 0$, there is no coordinate singularity for $v < 0$, a spacelike coordinate singularity developing for $v > 0$ according to

$$(-u)^{\frac{1}{c^2+1}} \simeq -B(1+c^2)v. \tag{8.25}$$

On the other hand, the location of the apparent horizon is given near $x = 0$ by

$$(-u)^{\frac{c^2+2}{c^2+1}} = -\frac{\tilde{r}'}{\rho'} \simeq \bar{B}(1 + (\gamma - \beta)x), \tag{8.26}$$

with $\bar{B} = 2B(c^2+2)/\Lambda(c^2+1) > 0$, $\beta = -\Lambda(c^2+1)/2$, $\gamma = -\Lambda(c^2+1)(4c^2+9)/2(2c^2+3)^2$, $0 < \gamma < \beta$ (the first order behaviour of ρ' has been computed

from Eq. (5.2)). This apparent horizon preexists the singularity (Fig. 6), its radius varying as

$$r_{AH} \simeq [\bar{B}(1+(\gamma-\beta)x)]^{1/k}(1+O(x^2)), \tag{8.27}$$

i.e. increasing with decreasing x, as suits a collapsing configuration. The behaviour of the apparent horizon in the region $x > 0$ is less clear. In particular, it would seem to be asymptotic to the null line $u = 0$ for some finite value $x = x_0$ (corresponding to $v \to -\infty$) , but at this point r_{AH} would diverge, in contradiction with our condition (A) for the validity of the linear perturbation analysis. A more refined calculation, including higher order perturbations, would be needed in order to clarify this issue.

Now we inquire whether the apparent horizon of the perturbed spacetime is regular. The non-constant part of the Ricci scalar (2.9) is, to first order, the product of three factors,

$$\begin{aligned} R+6\Lambda = \ & 4e^{-2\sigma}\phi_{,u}\phi_{,v} = 4e^{-2\nu}\,[1-2(-u)^{-\frac{k}{c^2+1}}\tilde{\sigma}]\cdot \\ & [-\frac{c}{c^2+1}+x\psi'+(-u)^{-\frac{k}{c^2+1}}(\frac{k}{c^2+1}\tilde{\phi}+x\tilde{\phi}')]\cdot \\ & [\psi'+(-u)^{-\frac{k}{c^2+1}}\tilde{\phi}']\,. \end{aligned} \tag{8.28}$$

Possible curvature singularities for $v = 0$ having being excluded by the analyticity condition (B2), perturbative curvature singularities can only arise from the divergence of any factor of (8.28) on the original null singularity $u = 0$ [4]. Leaving aside case b), where as mentioned above it is not clear whether the apparent horizon intersects the line $u = 0$, we concentrate on case a). The perturbative Ricci scalar (8.28) evaluated on the apparent horizon (8.19) is

$$\begin{aligned} R_{AH}+6\Lambda = \ & 4e^{-2\nu}\,[1+2\rho'\frac{\tilde{\sigma}}{\tilde{r}'}]\cdot \\ & [-\frac{c}{c^2+1}+x\psi'-\rho'\frac{\frac{k}{c^2+1}\tilde{\phi}+x\tilde{\phi}'}{\tilde{r}'}]\,[\psi'-\rho'\frac{\tilde{\phi}'}{\tilde{r}'}]\,. \end{aligned} \tag{8.29}$$

The last factor $\phi_{,v}$ diverges as x^{-1} when $x \to 0$, while the other factors go to finite limits, so that the apparent horizon will generically be singular at its onset $x = 0$, unless the product $e^{-2\sigma}\phi_{,u}$ accidentally vanishes,

$$e^{-2\sigma}\phi_{,u}|_{AH}(x=0) = 0. \tag{8.30}$$

[4]This again is due to a deficiency of the perturbative approach: the perturbative coordinate singularity due essentially to the vanishing of $e^{2\sigma}$ may, as mentioned above, be a more correct indicator of a curvature singularity.

We suggest that this condition of "regularity at birth" of the apparent horizon is the crucial boundary condition (D) which will determine the eigenvalue of k. Specifically, for $x \to 0$,

$$R_{AH} + 6\Lambda \propto \left(1 + \frac{c^2 + 3/2}{c^2 + 2}\right)\left(c - \frac{(2k - 1/2)(c^2 + 3/2)}{2c(c^2 + 2)}\right) x^{-1}. \qquad (8.31)$$

The first factor is positive, while the second factor vanishes for the value

$$k_0 = \frac{1}{4} + c^2 \frac{c^2 + 2}{c^2 + 3/2}. \qquad (8.32)$$

Unfortunately, it is easy to see that $k_0 < c^2 + 3/2$ for any c, so that the analyticity condition (B2) is not satisfied and consequently the Ricci scalar diverges on the line $v = 0$. Nevertheless we believe that the mechanism proposed here for the determination of k is basically correct, and that it is its implementation in the linear approximation which is responsible for the problem just mentioned. Indeed, a way to overcome this problem is to play with the ambiguities of the linear approximation and linearize the product $e^{-2\sigma}\phi_{,u}$ *before* evaluating it on the apparent horizon. Then, (8.31) is replaced by

$$R_{AH} + 6\Lambda \propto \left(c + \frac{c(c^2 + 3/2)}{c^2 + 2} - \frac{(2k - 1/2)(c^2 + 3/2)}{2c(c^2 + 2)}\right) x^{-1}, \qquad (8.33)$$

which vanishes for the value

$$k_1 = \frac{2c^4 + (15/4)c^2 + 3/8}{c^2 + 3/2}. \qquad (8.34)$$

This is smaller than $(2c^2+5/2)$ (the value (8.8) for $n = 2$), while the equation

$$k_1 = c^2 + 3/2 \qquad (8.35)$$

($n = 1$) has only one real solution $c^2 \simeq 1.045$, leading to $k \simeq 2.545$. While the linearization trick we have used to obtain this value is hard to justify, it is nevertheless suggestive that we have found a set of boundary conditions (B1), (B2), and (D) which determine a single value for the couple (c, k), i.e. a unique critical solution with a unique growing mode, and that this value $c^2 \simeq 1.045$ is very close to the value $c^2 = 1$ suggested as a critical value by the global analysis of our family of CSS solutions (Figs. 1 and 2). Let us also recall that Garfinkle [13] has found that the solution (2.10) is in good

agreement with the numerical results of [3] at an intermediate time for the special value $c^2 = 7/8 = 0.875$, which is also close to 1.

From the dependence (8.21) of the limiting horizon radius on the perturbation amplitude (proportional to the sole non-vanishing integration constant C or B), the critical exponent is $\gamma = 1/k$. For the preferred value $c^2 \approx 1$, this leads to $\gamma \approx 2/5$ [14] in case a) with $n = 1$, or $\gamma \approx 1/3$ in case b).

9 Discussion

The reason for writing this paper has been twofold. In the first part we have extended previous works [13, 14] motivated by the quest for the critical solution underlying the collapse of a massless and minimally coupled scalar field in 2+1 dimensional anti-de Sitter spacetime. In particular, we have constructed a family of exact solutions with CSS behavior near the singularity and AdS behavior at spatial infinity. This one-parameter (c) family of time-dependent solutions interpolates between the static vacuum solution ($c^2 = 0$) and the massive BTZ black hole ($c^2 = \infty$), with the critical value $c^2 = 1$ lying at the boundary between two different singularity types (point singularity, and line singularity).

These solutions served as the starting point for the subsequent linear perturbation analysis. A crucial question is that of the choice of the correct boundary conditions to be imposed along the null boundaries $u = 0$, $v = 0$, and the AdS boundary $x = x_1$. These boundary conditions have been "tested" in the simple case $c^2 = 0$, where the scalar field decouples and the linear perturbations describe exactly the small mass static BTZ black hole. In this respect, the essential difference with our previous work [14] is that we now recognize that the condition (C) that the perturbations do not blow up at the AdS boundary is always satisfied up to gauge transformations; we can then impose the natural smooth matching conditions (B1) (Eq. (6.8)) on the boundary $v = 0$, instead of the weaker condition $\tilde{r}(0) = 0$ used in [14] (resulting in significantly different behaviors of the coordinate singularity and apparent horizon). We also considered a new and crucial condition (D), stating the "regularity at birth" of the apparent horizon. Fulfilment of the selected boundary conditions then gives rise to two distinct growing modes a) (Eq. (8.8) with $n = 1$) and b) (Eq. (8.9)). The application of condition (D) is very suggestive because, together with the other boundary conditions it selects, for the mode a), a unique value for both the eigenvalue k and the parameter c.

The ensuing physical picture, along with the ambiguities inherent to the linear perturbations approximation, is also discussed. For the mode a) the singularity and the apparent horizon appear simultaneously on the null line $u = 0$ at the time $v = 0$ and evolve in the region $v > 0$ in a physically meaningful way. On the other hand, for the mode b) (which is the extension to $c^2 \neq 0$ of the single BTZ mode found for $c^2 = 0$), the singularity still appears for $v = 0$, but the apparent horizon seems to be eternal, as in the case of the static BTZ black hole. This formal solution cannot describe actual gravitational collapse with regular initial conditions. Indeed, we conjecture [22] that the solution b) is actually a linearization of the singular static solution (4.4). We would like to argue that ultimately the boundary conditions (apart from natural requirements on the consistency of the linear approximation) should be chosen so as to optimize the agreement of the perturbative picture with realistic gravitational collapse. On such a basis the mode b) should be excluded as unphysical. This would mean that there is exactly one growing mode a), showing that the background quasi-CSS solution is indeed the critical solution for gravitational collapse.

An objection against this view is that the quasi-CSS solutions presented in this paper do not have a regular timelike origin, while all near-critical collapse scenarios do evolve about a regular timelike origin. However we argue that a critical solution, which lies in the boundary surface separating different collapsing regimes, does not necessarily share with the generic collapsing configurations a regular central behavior, i.e. the critical solution can lie at the boundary of the set of generic collapsing solutions without belonging to the set. Indeed, further investigations currently under progress seem to show that certain perturbations of our quasi-CSS solutions do present a regular timelike center, and that, moreover, a subset of these perturbations correspond precisely to extended (to $\Lambda < 0$) Garfinkle solutions. This would suggest that our quasi-CSS solutions, with singular center, could be exact critical solutions, with near-critical solutions well described by the extended Garfinkle solutions.

We finally turn to the calculation of the critical exponent γ. Numerically, Pretorius and Choptuik [3] found γ in the range $1.15 < \gamma < 1.25$, while Husain and Olivier [5] estimated $\gamma \sim 0.81$. Choosing the critical value $c^2 = 1$ (which is also motivated by the application of boundary condition (D)), we obtain for the mode a) the value $\gamma = 0.4$, which differs significantly from both numerical values. We cannot at present give a reason for this disagreement (which recalls the disagreement between the numerical [1] and analytical [9] determinations of the critical exponent for scalar field collapse in 3+1 dimensions). Despite this shortcoming, we find it rather satisfactory

to have been able to reproduce from this solution a number of characteristic features of critical collapse and black hole formation.

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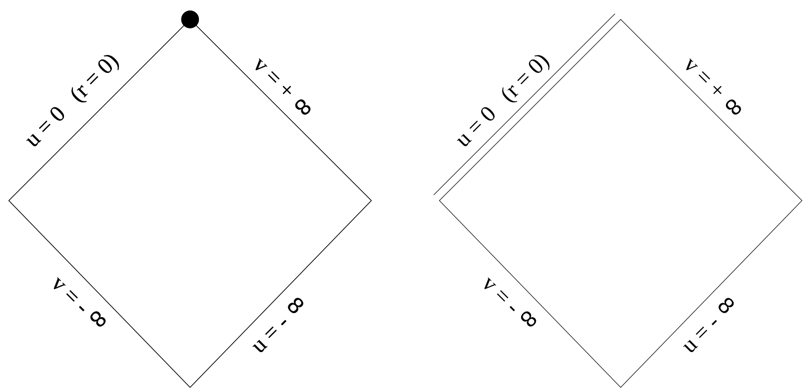

Figure 1: Penrose diagram of the CSS solutions eq. (4.3) ($\Lambda = 0$) a) for $c^2 \leq 1$, with a point singularity (dot) at $u = 0$, $v = +\infty$; b) for $c^2 > 1$, with a line singularity (double line) at $u = 0$.

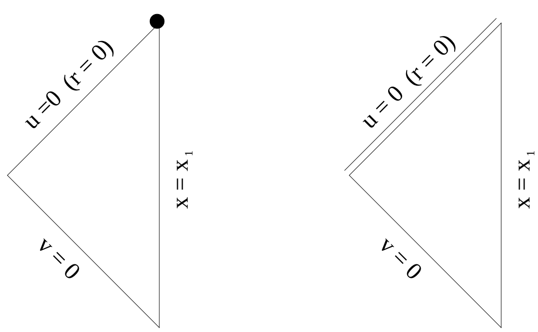

Figure 2: Penrose diagram of the extended quasi-CSS solutions ($\Lambda < 0$) a) for $c^2 \leq 1$; b) for $c^2 > 1$.

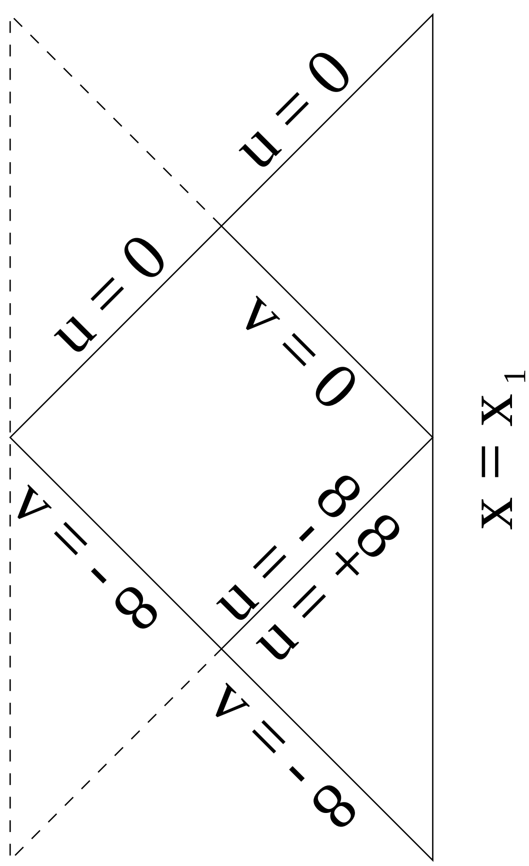

Figure 3: Penrose diagram of the vacuum solution (4.13). The analytic continuation from $u = -\infty$ to $u = +\infty$ is achieved with the map (7.11).

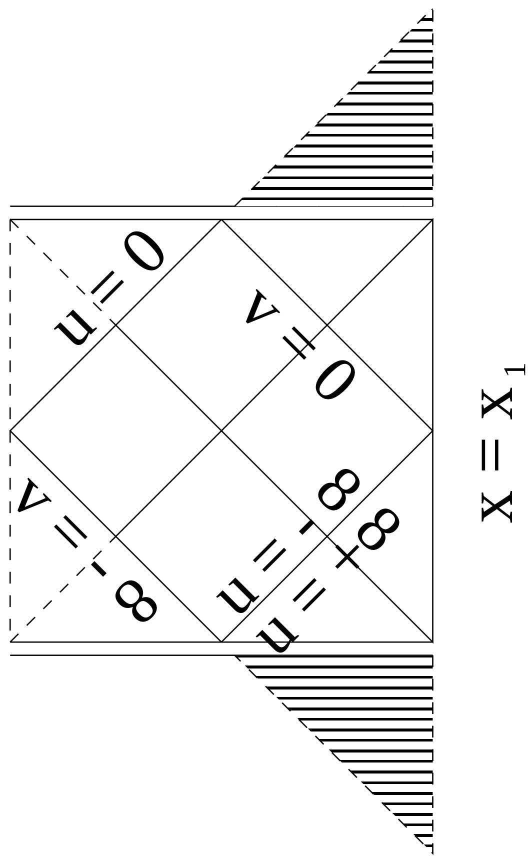

Figure 4: Penrose diagram of the BTZ black hole (7.8). The two spacelike central singularities (double lines) are shielded by two null horizons (the diagonals).

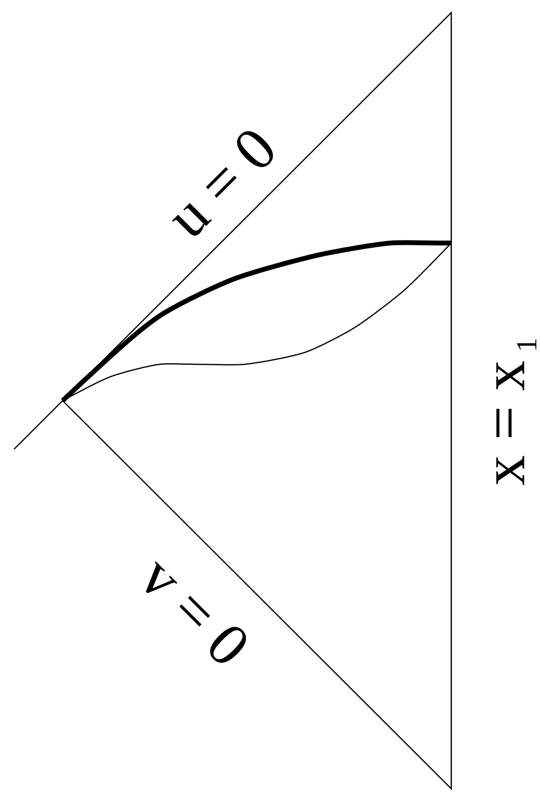

Figure 5: Scalar perturbations with $k = c^2 + 3/2$ (case a)). The spacelike coordinate singularity (thick curve) and apparent horizon appear simultaneously at v = 0.

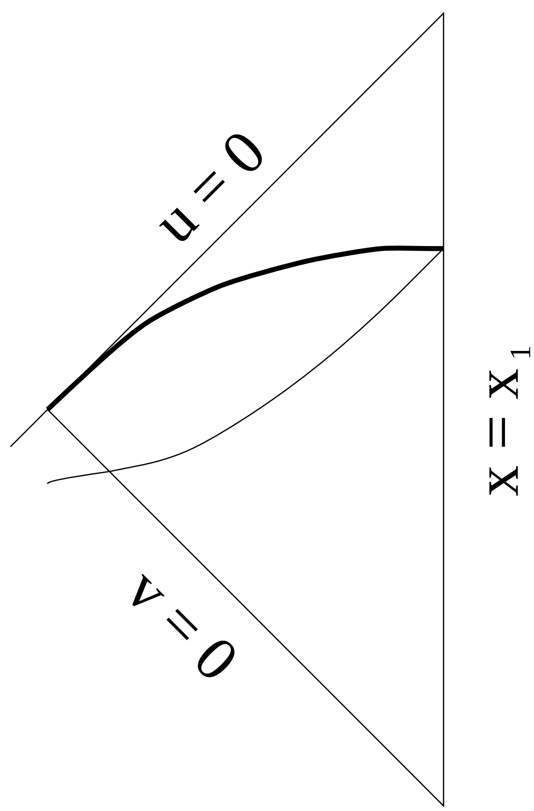

Figure 6: Scalar perturbations with $k = c^2 + 2$ (case b)). The apparent horizon prexists the spacelike singularity.